Annex 1
Basic equations of motion in fluid mechanics

1.1 Introduction

It is assumed that the reader of this book is familiar with the basic laws of fluid mechanics. Nevertheless some of these laws will be discussed in this annex to summarise material and to emphasize certain subjects which are important in the context of discharge measurement structures in open channels.

1.2 Equation of motion - Euler

In fluid mechanics we consider the motion of a fluid under the influence of forces acting upon it. Since these forces produce an unsteady motion, their study is essentially one of dynamics and must be based on Newton's second law of motion

\[ F = ma \]  

(A1.1)

where \( F \) is the force required to accelerate a certain mass (\( m \)) at a certain rate (\( a \)).

If we consider the motion of an elementary fluid particle (\( dx \ dy \ dz \)) with a constant mass-density (\( \rho \)), its mass (\( m \)) equals

\[ m = \rho \, dx \, dy \, dz \]  

(A1.2)

The following forces may act on this particle:

a. The normal pressures (\( P \)) exerted on the lateral faces of the elementary volume by the bordering fluid particles;
b. The mass forces, which include in the first place the gravitational force and in the second the power of attraction of the moon and the sun and the Coriolis force. These forces, acting on the mass (\( \rho \, dx \, dy \, dz \)) of the fluid particle, are represented together by their components in the X-, Y-, and Z-direction. It is common practice to express these components per unit of mass, and therefore as accelerations; for example, the gravitational force is expressed as the downward acceleration \( g \);
c. Friction. There are forces in a fluid which, due to friction, act as shear forces on the lateral faces of the elementary particle (\( dx \ dy \ dz \)). To prevent complications unnecessary in this context, the shear force is regarded as a mass force.

Gravitation and friction are the only mass forces we shall consider. If the fluid is in motion, these two forces acting on the particle (\( dx \ dy \ dz \)) do not have to be in equilibrium, but may result in an accelerating or decelerating force (pos. or neg.). This net force is named:

d. Net impressed force. This force equals the product of the mass of the particle and the acceleration due to the forces of pressure and mass not being in equilibrium.

The net impressed force may be resolved in the X-, Y-, and Z-direction.

If we assume that the pressure at a point is the same in all directions even when the fluid is in motion, and that the change of pressure intensity from point to point is
continuous over the elementary lengths dx, dy, and dz, we may define the normal pressures acting, at time t, on the elementary particle as indicated in Figure A1.1. Acting on the left-hand lateral face (X-direction) is a force

\[ + \left( P - \frac{1}{2} \frac{\partial P}{\partial x} \ dx \right) dy \ dz \]

while on the right-hand face is a force

\[ -(P + \frac{1}{2} \frac{\partial P}{\partial x} \ dx) dy \ dz \]

The resulting normal pressure on the elementary fluid particle in the X-direction equals

\[ - \frac{\partial P}{\partial x} \ dx \ dy \ dz \] (A1.3)

The resultant of the combined mass forces in the X-direction equals

\[ \rho \ dx \ dy \ dz \ k_x \]

where \( k_x \) is the acceleration due to gravitation and friction in the X-direction. Hence in the X-direction, normal pressure and the combined mass forces on the elementary particle result in a total force

\[ F_x = - \frac{\partial P}{\partial x} \ dx \ dy \ dz + k_x \rho \ dx \ dy \ dz \] (A1.4)
Similarly, for the forces acting on the mass (ρ dx dy dz) in the Y- and Z-direction, we may write

\[ F_y = -\frac{\partial P}{\partial y} \rho dx dy dz + k_y \rho dx dy dz \] (A1.5)

and

\[ F_z = -\frac{\partial P}{\partial z} \rho dx dy dz + k_z \rho dx dy dz \] (A1.6)

The reader should note that in the above equations \( k_x \), \( k_y \), and \( k_z \) have the dimension of an acceleration.

In a moving liquid the velocity varies with both position and time (Figure A1.2). Hence:

\[ v = f(x, y, z, t) \] (A1.7)

and as such

\[ v_x = f_x(x, y, z, t) \]
\[ v_y = f_y(x, y, z, t) \]

and

\[ v_z = f_z(x, y, z, t) \]

If we consider the X-direction first, we may write that at the time (t + dt) and at the point (x + dx, y + dy, z + dz) there is a velocity component in the X-direction which equals \( v_x + dv_x \).
The total differential of $v_x$ is equal to

$$dv_x = \frac{\partial v_x}{\partial t} dt + \frac{\partial v_x}{\partial x} dx + \frac{\partial v_x}{\partial y} dy + \frac{\partial v_x}{\partial z} dz$$  \hspace{1cm} (A1.8)$$

In Figure A1.3 we follow a moving fluid particle over a time $dt$, and see it moving along a pathline from point $(x, y, z)$ towards point $(x + dx, y + dy, z + dz)$ where it arrives with another velocity component $(v_x + dv_x)$. The acceleration of the fluid particle in the X-direction consequently equals

$$a_x = \frac{dv_x}{dt}$$  \hspace{1cm} (A1.9)$$

while the elementary variations in time and space equal

$$dx = v_x dt$$  \hspace{1cm} (A1.10)$$
$$dy = v_y dt$$  \hspace{1cm} (A1.11)$$
$$dz = v_z dt$$  \hspace{1cm} (A1.12)$$

Equation A1.8, which is valid for a general flow pattern, also applies to a moving fluid particle as shown in Figure A1.3, so that Equations A1.10 to A1.12 may be substituted into Equation A1.8, giving

$$dv_x = \frac{\partial v_x}{\partial t} dt + \frac{\partial v_x}{\partial x} v_x dt + \frac{\partial v_x}{\partial y} v_y dt + \frac{\partial v_x}{\partial z} v_z dt$$  \hspace{1cm} (A1.13)$$

and after substitution of Equation A1.9

$$a_x = \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z$$  \hspace{1cm} (A1.14)$$
and similarly

\[ a_y = \frac{d v_y}{dt} = \frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial x} v_x + \frac{\partial v_y}{\partial y} v_y + \frac{\partial v_y}{\partial z} v_z \]  

\[ a_z = \frac{d v_z}{dt} = \frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial x} v_x + \frac{\partial v_z}{\partial y} v_y + \frac{\partial v_z}{\partial z} v_z \]  

(A1.15)  

(A1.16)


\[-\frac{\partial P}{\partial x} dx dy dz + k_x \rho dx dy dz = \rho dx dy dz \left[ \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] \]

or

\[ \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z = -\frac{1}{\rho} \frac{\partial P}{\partial x} + k_x \]  

(A1.17)

In the same manner we find for the Y- and Z-direction

\[ \frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial x} v_x + \frac{\partial v_y}{\partial y} v_y + \frac{\partial v_y}{\partial z} v_z = -\frac{1}{\rho} \frac{\partial P}{\partial y} + k_y \]  

(A1.18)

\[ \frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial x} v_x + \frac{\partial v_z}{\partial y} v_y + \frac{\partial v_z}{\partial z} v_z = -\frac{1}{\rho} \frac{\partial P}{\partial z} + k_z \]  

(A1.19)

These are the Euler equations of motion, which have been derived for the general case of unsteady non-uniform flow and for an arbitrary Cartesian coordinate system.

An important simplification of these equations may be obtained by selecting a coordinate system whose origin coincides with the observed moving fluid particle (point P). The directions of the three axes are chosen as follows:

- s-direction: the direction of the velocity vector at point P, at time t. As defined, this vector coincides with the tangent to the streamline at P at time t (\(v_s = v\)).
- n-direction: the principal normal direction towards the centre of curvature of the streamline at point P at time t. As defined, both the s- and n-direction lie in the osculating plane.
- m-direction: the binormal direction perpendicular to the osculating plane at P at time t (see also Chapter 1).

If we assume that a fluid particle is passing through point P at time t with a velocity \(v\), the Eulerian equations of motion can be written as:

\[ \frac{\partial v_s}{\partial t} + \frac{\partial v_s}{\partial s} v_s + \frac{\partial v_s}{\partial n} v_n + \frac{\partial v_s}{\partial m} v_m = -\frac{1}{\rho} \frac{\partial P}{\partial s} + k_s \]  

(A1.20)

\[ \frac{\partial v_n}{\partial t} + \frac{\partial v_n}{\partial s} v_s + \frac{\partial v_n}{\partial n} v_n + \frac{\partial v_n}{\partial m} v_m = -\frac{1}{\rho} \frac{\partial P}{\partial n} + k_n \]  

(A1.21)

\[ \frac{\partial v_m}{\partial t} + \frac{\partial v_m}{\partial s} v_s + \frac{\partial v_m}{\partial n} v_n + \frac{\partial v_m}{\partial m} v_m = -\frac{1}{\rho} \frac{\partial P}{\partial m} + k_m \]  

(A1.22)

Due to the selection of the coordinate system, there is no velocity perpendicular to the s-direction; thus

\[ v_n = 0 \quad \text{and} \quad v_m = 0 \]  

(A1.23)
Therefore the equations of motion may be simplified to

\[
\frac{\partial v_s}{\partial t} + \frac{\partial v_s}{\partial s} v_s = -\frac{1}{\rho} \frac{\partial P}{\partial s} + k_s
\]

(A1.24)

\[
\frac{\partial v_n}{\partial t} + \frac{\partial v_n}{\partial s} v_s = -\frac{1}{\rho} \frac{\partial P}{\partial n} + k_n
\]

(A1.25)

\[
\frac{\partial v_m}{\partial t} + \frac{\partial v_m}{\partial s} v_s = -\frac{1}{\rho} \frac{\partial P}{\partial m} + k_m
\]

(A1.26)

Since the streamline at both sides of P is situated over an elementary length in the osculating plane, the variation of \(v_m\) in the \(s\)-direction equals zero. Hence, in Equation A1.26

\[
\frac{\partial v_m}{\partial s} = 0
\]

(A1.27)

In Figure A1.4 an elementary section of the streamline at point P at time \(t\) is shown in the osculating plane. It can be seen that

\[
\tan d\beta = \frac{\frac{\partial v_s}{\partial s} ds}{v_s + \frac{\partial v_s}{\partial s} ds} = \frac{ds}{r}
\]

(A1.28)

or

\[
\frac{\partial v_n}{\partial s} = \frac{v_s + \frac{\partial v_s}{\partial s} ds}{r}
\]

(A1.29)

In the latter equation, however, \(\frac{\partial v_s}{\partial s}\) \(ds\) is infinitely small compared with \(v_s\); thus we may rewrite Equation A1.29 as

\[
\frac{\partial v_n}{\partial s} = \frac{v_s}{r}
\]

(A1.30)

or

\[
\frac{\partial v_n}{\partial s} v_s = \frac{v_s^2}{r}
\]

(A1.31)

Substitution of Equations A1.27 and A1.31 into Equations A1.26 and A1.25 respectively gives Euler's equations of motion as follows

\[
\frac{\partial v_s}{\partial t} + \frac{\partial v_s}{\partial s} v_s = -\frac{1}{\rho} \frac{\partial P}{\partial s} + k_s
\]

(A1.32)

\[
\frac{\partial v_n}{\partial t} + \frac{v_s^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial s} + k_n
\]

(A1.33)
These equations of motion are valid for both unsteady and non-uniform flow. Hereafter, we shall confine our attention to steady flow, in which case all terms $\partial \ldots / \partial t$ equal zero.

Equations A1.32, A1.33, and A1.34 are of little use in direct applications, and they tend to repel engineers by the presence of partial derivative signs; however, they help one's understanding of certain basic equations, which will be dealt with below.

1.3 Equation of motion in the s-direction

If we follow a streamline (in the s-direction) we may write $v_s = v$, and the partial derivatives can be replaced by normal derivatives because $s$ is the only dependent variable. (Thus $\partial$ changes into $d$). Accordingly, Equation A1.32 reads for steady flow

$$\frac{\partial v_m}{\partial t} = \frac{-1}{\rho} \frac{\partial P}{\partial s} + k_s$$

(A1.34)

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$$\frac{dv}{ds} = \frac{-1}{\rho} \frac{\partial P}{\partial s} + k_s$$

(A1.35)

where $k_s$ is the acceleration due to gravity and friction. We now define the negative Z-direction as the direction of gravity. The weight of the fluid particle is $- \rho g ds dn dm$ of which the component in the s-direction is

$$- \rho g ds dn dm \frac{dz}{ds}$$

and per unit of mass

$$\frac{- \rho g ds dn dm \frac{dz}{ds}}{\rho ds dn dm} = -g \frac{dz}{ds}$$

(A1.36)
The force due to friction acting on the fluid particle in the negative s-direction equals per unit of mass

\[-w = \frac{-W}{\rho \, ds \, dn \, dm}\]  \hspace{1cm} (A1.37)

The acceleration due to the combined mass-forces ($k_s$) acting in the s-direction accordingly equals

\[k_s = -w - g \frac{dz}{ds}\]  \hspace{1cm} (A1.38)

Substitution of this equation into Equation A1.35 gives

\[\frac{dv}{ds} = -\frac{1}{\rho} \frac{dP}{ds} - g \frac{dz}{ds} - w\]  \hspace{1cm} (A1.39)

or

\[\rho \frac{v}{ds} \frac{dv}{ds} + \frac{dP}{ds} + \rho g \frac{dz}{ds} = -\rho \, w\]  \hspace{1cm} (A1.40)

or

\[d \left(\frac{1}{2} \rho \, v^2 + P + \rho g z\right) = -\rho \, w \, ds\]  \hspace{1cm} (A1.41)

The latter equation indicates the dissipation of energy per unit of volume due to local
friction. If, however, the decelerating effect of friction is neglected, Equation A1.41 becomes

\[ \frac{d}{ds} \left( \frac{1}{2} \rho v^2 + P + \rho gz \right) = 0 \]  

(A1.42)

Hence

\[ \frac{1}{2} \rho v^2 + P + \rho gz = \text{constant} \]  

(A1.43)

where

\[ \frac{1}{2} \rho v^2 = \text{kinetic energy per unit of volume} \]
\[ \rho gz = \text{potential energy per unit of volume} \]
\[ P = \text{pressure energy per unit of volume} \]

If Equation A1.43 is divided by \( \rho g \), an equation in terms of head is obtained, which reads

\[ \frac{v^2}{2g} + \frac{P}{\rho g} + z = \text{constant} = H \]

(A1.44)

where

\[ \frac{v^2}{2g} = \text{the velocity head} \]
\[ \frac{P}{\rho g} = \text{the pressure head} \]
\[ z = \text{the elevation head} \]
\[ \frac{P}{\rho g} + z = \text{the piezometric head} \]
\[ H = \text{the total energy head} \]

The last three heads all refer to the same reference level (see Figure 1.3, Chapter I).

The Equations A1.43 and A1.44 are alternative forms of the well-known Bernoulli equation, and are valid only if we consider the movement of an elementary fluid particle along a streamline under steady flow conditions (pathline) with the mass-density (\( \rho \)) constant, and that energy losses can be neglected.

1.4 Piezometric gradient in the n-direction

The equation of motion in the n-direction reads for steady flow (see Equation A1.33)

\[ \frac{v^2}{r} = - \frac{1}{\rho} \frac{dP}{dn} + k_n \]  

(A1.45)

Above, the \( \partial \) has been replaced by \( d \) since \( n \) is the only independent variable. The term \( v^2/r \) equals the force per unit of mass acting on a fluid particle which follows a curved path with radius \( r \) at a velocity \( v \) (centripetal acceleration). In Equation A1.45, \( k_n \) is the acceleration due to gravity and friction in the n-direction. Since \( v_n = 0 \), there is no friction component. Analogous to its component in the direction of flow here the component due to gravitation can be shown to be

\[ k_n = - g \frac{dz}{dn} \]  

(A1.46)
Substitution into Equation A1.45 yields
\[
\frac{v^2}{r} = -\frac{1}{\rho} \frac{dP}{dn} - g \frac{dz}{dn}
\]  
(A1.47)
which, after division by g, may be written as
\[
d \left( \frac{P}{\rho g} + z \right) = -\frac{v^2}{gr} \frac{dn}{dn}
\]  
(A1.48)
After integration of this equation from point 1 to point 2 in the n-direction we obtain the following equation for the change of piezometric head in the n-direction
\[
\left( \frac{P}{\rho g} + z \right)_1 - \left( \frac{P}{\rho g} + z \right)_2 = \frac{1}{g} \int_1^2 \frac{v^2}{r} \frac{dn}{dn}
\]  
(A1.49)
where \((P/\rho g + z)\) equals the piezometric head at point 1 and 2 respectively and
\[
\frac{1}{g} \int_1^2 \frac{v^2}{r} \frac{dn}{dn}
\]
is the loss of piezometric head due to curvature of the streamlines.

1.5 Hydrostatic pressure distribution in the m-direction
Perpendicular to the osculating plane, the equation of motion, according to Euler, reads for steady flow
\[
-\frac{1}{\rho} \frac{dP}{dm} + k_m = 0
\]  
(A1.50)
Since there is no velocity component perpendicular to the osculating plane \( (v_m = 0) \), there is no friction either. The component of the acceleration due to gravity in the m-direction is obtained as before, so that

\[
k_m = -g \frac{dz}{dm}
\]  

(A1.51)

Substitution of this acceleration in the equation of motion (Equation A1.50) gives

\[
\frac{1}{\rho} \frac{dP}{dm} - g \frac{dz}{dm} = 0
\]  

which may be written as

\[
\frac{d}{dm} \left( \frac{P}{\rho g} + z \right) = 0
\]  

(A1.53)

It follows from this equation that the piezometric head in the m-direction is

\[
\frac{P}{\rho g} + z = \text{constant}
\]  

(A1.54)

irrespective of the curvature of the streamlines. In other words, perpendicular to the osculating plane, there is a hydrostatic pressure distribution.